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ON THE ALMOST CHEBYSHEVIAN APPROXIMATION TO CERTAIN OPERATORS OF THE HEREDITARY THEORY OF ELASTICITY*

E.S. SINAISKII

Resolvent hereditary operator generated by an integral, Volterra type operator with the Abel /1/ or Rzhanitsyn /2/ kernel is approximated on an arbitrary, finite time interval, by a polynomial in fractional powers of a variable with an exponential co-factor, using the method developed in /3,4/. The approximation obtained from all polynomials of the given type offers, firstly, a smallest error in the defining equation, and secondly, it approaches asymptotically with increasing order the Chebyshev polynomials of the best uniform approximation to the function on the segment. The estimation of the approximation obtained shows that the error decreases with increasing order of approximation at least as rapidly as the geometrical progression.

Application of the Volterra principle together with the algebra of the resolvent operators /1,5/ makes it necessary to construct, at the last stage of solving the problem of hereditary theory of elasticity, the heredity operator. This is usually done by numerical methods, since the quadratures appearing in the process cannot, as a rule, be expressed in their final form in terms of the elementary functions. The known approximations with satisfactorily determinable errors have asymptotic character, i.e. are suitable when the time variable is sufficiently large or sufficiently small. The intermediate time interval however, over which the relaxation and aftereffect processes still continue to develop, are also of interest.

1. Let $R_\alpha^*(\lambda, \beta)$ be an integral, Volterra-type operator

$$R_\alpha^*(\lambda, \beta) f(t) \equiv \int_0^t R_\alpha(\lambda, \beta; t-s) f(s)\, ds \tag{1.1}$$

$$R_\alpha(\lambda, \beta; t) = e^{\lambda t} \vartheta_{\alpha-1}(\beta; t) = e^{\lambda t} \sum_{n=1}^{\infty} \frac{\beta^{n-1} t^{n\alpha-1}}{\Gamma(n\alpha)}, \qquad \lambda \leqslant 0,\ \beta \leqslant 0,\ 0 < \alpha \leqslant 1 \tag{1.2}$$

When $\lambda = 0$, the function (1.2) becomes a fractional power Rabotnov exponent /1/, at $\beta = 0$ it becomes a Rzhanitsyn kernel /2/, at $\lambda = \beta = 0$ an Abel kernel and at $\alpha = 1$ a normal exponent. Let us consider the problem of approximating a convolution of the type (1.1) on an arbitrary finite time interval $[0, t_1]$

$$\begin{aligned} &\beta R_\alpha^*(\lambda, \beta) f_1(t) = \chi R_\alpha^*(\mu, \chi) f(\theta) \equiv u(\theta) \\ &t = \theta t_1,\ 0 \leqslant \theta \leqslant 1,\ f(\theta) = f_1(\theta t_1),\ \chi = \beta t_1^\alpha < 0,\ \mu = \lambda t_1 \leqslant 0,\ 0 < \alpha < 1 \end{aligned} \tag{1.3}$$

We write the function $f(\theta)$ in the form

$$f(\theta) = e^{\mu\theta} \sum_{j=0}^{m} f_j \theta^{\alpha j + \nu}, \quad m \geqslant 0 \tag{1.4}$$

The parameter $\nu \geqslant 0$ is arbitrary, while α and μ are the same as in (1.3). The approximation of a sufficiently smooth function (in this case of $f(\theta)\exp(-\mu\theta)$) by a power polynomial, represents a traditional and well researched problem of the function approximation theory /6/.

Therefore the particular manner of representing the function (1.4) should not be regarded as an excessively rigid constraint.

The resolvent operator $R_\alpha^*(\mu, \chi)$ is connected with its generating operator $R_\alpha^*(\mu, 0)$ by the relations /5/ (I is a unit operator)

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$$I + \chi R_\alpha^*(\mu,\chi) = [I - \chi R_\alpha^*(\mu, 0)]^{-1}; \quad R_\alpha^*(\mu,\chi) = \frac{R_\alpha^*(\mu,0)}{I-\chi R_\alpha^*(\mu,0)} \tag{1.5}$$

Taking into account (1.5) we transform (1.3) into a Volterra equation for the function $u(\theta)$

$$[I - \chi R_\alpha^*(\mu, 0)]\, u(\theta) = \chi R_\alpha^*(\mu, 0)\, f(\theta) \tag{1.6}$$

Setting

$$u(\theta) = \theta^\nu \quad e^{\mu\theta} B(\theta) \tag{1.7}$$

we seek the unknown function $B(\theta)$ in the form of a polynomial

$$B_n(\theta^\alpha) = \sum_{j=0}^{n} b_j \theta^{\alpha j}, \quad n \geqslant m \tag{1.8}$$

According to (1.1) and (1.2) we have

$$R_\alpha^*(\mu, 0)\, \theta^{\alpha j+\nu} e^{\mu\theta} = \Omega_{j+1}^j e^{\mu\theta} \theta^{\alpha(j+1)-\nu}, \quad \Omega_k^j = \frac{\Gamma(\alpha j+\nu+1)}{\Gamma(\alpha k+\nu+1)} \tag{1.9}$$

i.e. action of the operator $R_\alpha^*(\mu, 0)$ on the function in question yields the same type function. This makes it possible to construct, after substituting (1.4) and (1.8) into (1.6), a system of $n+2$ linear algebraic equations in $n+1$ unknowns b_j, using the method of undetermined coefficients. The overdefining does not allow us to obtain an exact solution of (1.6) in the form chosen. This can however be attained whrn the equation is "corrected" by introducing into it a suitably chosen error /3,4/. We replace (1.6) by

$$[I - \chi R_\alpha^*(\mu, 0)]\, B_n(\theta^\alpha)\, e^{\mu\theta}\theta^\nu = \chi R_\alpha^*(\mu, 0) f(\theta) + \tau e^{\mu\theta}\theta^\nu G_{n+1}(\theta^\alpha) \tag{1.10}$$

$$G_{n+1}(\theta^\alpha) = \sum_{j=0}^{n+1} g_j \theta^{\alpha j} \tag{1.11}$$

where τ is an unknown parameter introduced in order to remove the overdefinition mentioned above. Assuming that a good approximation can be obtained with help of the polynomial G_{n+1} evenly oscillating over the whole interval of the variable change, and we have ($T_n^0(x)$ is the displaced Chebyshev polynomial /4/)

$$G_{n+1}(\theta^\alpha) = T_{n+1}^0(\theta^\alpha) = \sum_{j=0}^{n+1} c_j^{n+1} \theta^{\alpha j} \tag{1.12}$$

$$T_n^0(x) = T_n(2x-1) = \cos n \arccos(2x-1) \tag{1.13}$$

By virtue of (1.4), (1.8), (1.9) and (1.12) we conclude, that equation (1.10) leads to the system

$$b_j = \chi\Omega_j^{j-1}(b_{j-1} + f_{j-1}) + \tau c_j^{n+1}, \quad j = 0, 1, 2, \ldots, n+1 \tag{1.14}$$

with $f_{-1} = b_{-1} = b_{n+1} = 0$ and $f_j = 0$ if $j > m$. The solution of the system (1.14) is

$$b_i = \sum_{j=0}^{i} \chi^{i-j}\Omega_i^j (f_j + \tau c_j^{n+1}) - f_i, \quad i = 0, 1, 2, \ldots, n \tag{1.15}$$

$$\tau = -\Big[\sum_{j=0}^{m} \chi^{-j}\Gamma(\alpha j+\nu+1) f_j\Big]\Big[\sum_{j=0}^{n+1} \chi^{-j}\Gamma(\alpha j+\nu+1) c_j^{n+1}\Big]^{-1} \tag{1.16}$$

and we denote the polynomials $B_n(\theta^\alpha)$ with coefficients (1.15) by $B_n^0(\theta^\alpha)$.

Theorem 1. Out of all possible polynomials $B_n(\theta^\alpha)$ of the form (1.8) and of degree not exceeding n, the polynomials $B_n^0(\theta^\alpha)$ contribute the least error to the equations (1.6). We have the following estimate for any natural $n \geqslant m$ (the constants $q < 1$ and Λ are independent of n)

$$\| B(\theta) - B_n(\alpha^\alpha) \| \leqslant 2|\tau| \leqslant 2^{4-\nu}\Lambda q^{n+1} \quad (\|\varphi(\theta)\| = \max_{0\leqslant\theta\leqslant 1} |\varphi(\theta)|) \tag{1.17}$$

$$q = (\xi + \sqrt{\xi^2 - 1})^{-1}, \quad \xi = 1 + 2^{1+\alpha}|\chi|^{-1}, \tag{1.18}$$

$$\Lambda = \Big|\sum_{j=0}^{m} \chi^{-j}\Gamma(\alpha j+\nu+1) f_j\Big|$$

Proof. Let $G_{n+1}(\theta^\alpha)$ be any polynomial of the form (1.11) causing an error in (1.10). The solution of (1.10) is represented by the polynomial $B_n(\theta^\alpha)$ (1.8), the coefficients of which are obtained from the formulas analogous to (1.15) and (1.16) with c_j^{n+1} replaced by g_j. All possible polynomials $B_n(\theta^\alpha)$ approaching $B(\theta)$ can be obtained with help of the above method, choosing $G_{n+1}(\theta^\alpha)$ arbitrarily but in such a manner that the denominator in the formula for τ of the form (1.16) does not vanish, i.e. under the condition

$$W = \left|\sum_{j=0}^{n+1} \chi^{-j}\Gamma(\alpha j + \nu + 1) g_j\right| \neq 0 \tag{1.19}$$

Since the factor of τ in (1.10) is undefined, we can assume without loss of generality that

$$\max_{0\leqslant\theta\leqslant 1} |G_{n+1}(\theta^\alpha)| = 1 \tag{1.20}$$

All coefficients of such a polynomial satisfy the inequalities /6/

$$|g_j| \leqslant |c_j^{n+1}|, j = 0, 1, 2, \ldots, n+1 \tag{1.21}$$

where c_j^{n+1} are the coefficients of the Chebyshev polynomial (1.13). Taking into account (1.21) and remembering that the product $\chi^{-j}c_j^{n+1}$ retains its sign when j is varied, we obtain

$$W \leqslant \left|\sum_{j=0}^{n+1} \chi^{-j}\Gamma(\alpha j + \nu + 1) c_j^{n+1}\right| \equiv W_0 \tag{1.22}$$

Thus, out of all polynomials $G_{n+1}(\theta^\alpha)$ satisfying the conditions (1.19) and (1.20) the polynomial $T_{n+1}^0(\theta^\alpha)$ imparts the highest value to the quantity W. Since the numerator in (1.16) is independent of the choice of the polynomial $G_{n+1}(\theta^\alpha)$, we also find that the parameter τ attains its smallest value in module and hence the smallest maximum of the modulus of the error in (1.10).

To obtain the estimate (1.17) we consider the function

$$\begin{gathered}\Phi(x) = \ln\Gamma(1 + \nu + x) - \ln\Gamma(1 + a) - (x - a + \nu)\ln a \\ x > -1 - \nu, \quad \nu \geqslant 0, \ a > 1\end{gathered} \tag{1.23}$$

The function $\Phi(x)$ is convex in the downward direction due to the logarithmic convexity of the gamma function /7/. Since $\Phi(a - \nu - 1) = \Phi(a - \nu) = 0$, it follows that the unique minimum γ_0 of this function appears on the interval $(a - \nu - 1,\ a - \nu)$. Choosing an arbitrary $\gamma < \gamma_0$ for all $x > -1 - \nu$, we obtain $\Phi(x) > \gamma$ or

$$\Gamma(1 + \nu + x) > e^\gamma\Gamma(1 + a)\, a^{\nu-a+x} = \delta a^x, \quad \delta = e^\gamma\Gamma(1 + a)\cdot a^{\nu-a} = \text{const}$$

Setting $a = 2$ we have $\Phi(x) > \gamma = -\ln 2$ /7/ for $x \in (1 - \nu, 2 - \nu)$. Moreover, $\delta = 2^{\nu-2}$ and hence $\Gamma(1 + \nu + x) > 2^{\nu-2+x}$ for all $x > -1 - \nu$. Taking into account this inequality together with the formulas (1.12) and (1.13) and remembering that $|c_j^{n+1}| = (-1)^{n+j+1}c_j^{n+1}$, we obtain

$$W_0 = \sum_{j=0}^{n+1} |\chi|^{-j}\Gamma(\alpha j + \nu + 1)|c_j^{n+1}| > 2^{\nu-2}\sum_{j=0}^{n+1} |\chi|^{-j} 2^{\alpha j}|c_j^{n+1}| = 2^{\nu-2}T_{n+1}(\xi), \quad \xi = 1 + 2^{1+\alpha}|\chi|^{-1}$$

from which, using the representation /6/

$$T_{n+1}(\xi) = {}^1/_2\{[\xi + (\xi^2 - 1)^{1/2}]^{n+1} + [\xi - (\xi^2 - 1]^{1/2}]^{n+1}\}$$

we obtain

$$W_0 > 2^{\nu-2}T_{n+1}(\xi) > 2^{\nu-3}[\xi + (\xi^2 - 1)^{1/2}]^{n+1}$$

Using the notation of (1.18) and (1.22) we obtain from (1.16) the inequality

$$|\tau| < 2^{3-\nu}\Lambda q^{n+1}, \ q < 1 \tag{1.24}$$

while (1.6) and (1.10) yield by virtue of (1.12), (1.7), (1.5) and (1.2),

$$B(\theta) - B_n^\circ(\theta^\alpha) = -\tau\left[T_{n+1}^0(\theta^\alpha) + \chi\int_0^\theta \vartheta_{\alpha-1}(\chi; \theta - s)(s/\theta)^\nu T_{n+1}^0(s^\alpha)\, ds\right] \tag{1.25}$$

Remembering that /1/

$$\int_0^\infty \vartheta_{\alpha-1}(\chi; s)\, ds = |\chi|^{-1}; \quad |T_{n+1}^0(\theta^\alpha)| \leqslant 1, \quad \theta \in [0,1]$$

from (1.25) and (1.24) we obtain (1.17), and this proves Theorem 1.

It is also evident that the following inequality holds for the function $u(\theta)$ sought:

$$\| u(\theta) - e^{\mu\theta}\theta^\nu B_n^0(\theta^\alpha)\| \leqslant \| B(\theta) - B_n^0(\theta^\alpha)\| \leqslant 2^{4-\nu}\Lambda q^{n+1} \tag{1.26}$$

Let e.g. $\alpha = 0.3, \lambda = 0, \chi = \beta t_1^\alpha = -2$. Then according to (1.18) $q = 0.237$ and in conformity with (1.26) we find that increasing the order of approximation by one reduces the error by more than four times. Thus, for the given values of the parameters we find, for the function $f(\theta) = 1$ ($\nu = 0$, $f_0 = 1$, $f_j = 0$ when $j > 0$) using the formula (1.16), that for $n = 5, 6, 7$ we have $|\tau| = 7.05 \cdot 10^{-4}$, $1.69 \cdot 10^{-4}$, $4.04 \cdot 10^{-5}$ respectively. For the function $f(\theta) = \theta^2 = \theta^{0.3+1.7}$ ($\nu = 1.7$, $f_1 = 1$, $f_j = 0$ when $j \neq 1$) and $n = 5$, we have $|\tau| = 1.55 \cdot 10^{-4}$. This means that a polynomial of the type

$$\sum_{j=0}^{5} b_j \theta^{0.3j+1.7}$$

with the coefficients given by the formulas (1.15) approaches uniformly over the whole interval in question, the convolution $\chi R_\alpha^* (0, \chi)\theta^2$ with an error not exceeding $3.1 \cdot 10^{-4}$.

2. The Chebyshev or the best uniform approximation of the continuous function $B(\theta)$ is determined, with help of all possible polynomials of degree not higher than n of the form (1.8), using the element B_n^∇, on which the best approximation to $E_n(B)$ /6/ is attained

$$\| B - B_n^\nabla \| = E_n(B), \quad E_n(B) \equiv \inf_{B_n} \| B(\theta) - B_n(\theta^\alpha) \| \tag{2.1}$$

The following theorem establishes the relation connecting the approximation obtained with help of the above polynomials $B_n^0(\theta^\alpha)$, and $E_n(B)$.

Theorem 2. The polynomials $B_n^0(\theta^\alpha)$ (1.8) with coefficients determined by the formulas (1.15) and (1.16) have the following property on the interval $0 \leqslant \theta \leqslant 1$:

$$\| B(\theta) - B_n^0(\theta^\alpha) \| = [1 + O(n^{-\alpha})] E_n(B), \quad n \to \infty \tag{2.2}$$

Proof. Performing the substitution $s^\alpha = x$, $\theta^\alpha = y$ we transform the integral term in (1.25) into the integral which can be estimated with help of the lemma proved in /8/. When $n \to \infty$, the integral in question decreases as quantity $O(n^{-\alpha})$, then we obtain from (1.25)

$$E_n(B) \leqslant \| B(\theta) - B_n^0(\theta^\alpha) \| \leqslant [1 + O(n^{-\alpha})] \, |\tau| \tag{2.3}$$

At sufficiently large n the sign of the right-hand side of (1.25) is determined by the sign of $T_{n+1}^0(\theta^\alpha)$ the extremal values of which, equal to ± 1, are attained with the sign alternating at $n+2$ consecutive points of the segment $[0, 1]$. Therefore by virtue of the Vallée - Poussin theorem /6/ we obtain

$$E_n(B) \geqslant |\tau| \, [1 - O(n^{-\alpha})], \quad n \to \infty$$

which yields the estimate

$$|\tau| \leqslant [1 + O(n^{-\alpha})] E_n(B), \quad n \to \infty \tag{2.4}$$

The relations (2.3) and (2.4) together yield (2.2). The theorem implies that as $n \to \infty$, the polynomials $B_n^0(\theta^\alpha)$ tend asymptotically to the element of the best uniform approximation B_n^∇.

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